Hydrodynamics and Chaos in Quantum Matter

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Motivation

• A long-standing problem has been to characterize hydrodynamic transport (charge, energy dynamics) in strongly coupled systems.

• Remarkably over the last few years evidence has emerged to suggest that this is related to scrambling and many-body chaos.
Part I

Diffusion and the Butterfly Velocity

MAB -1603.08510 (PRL)
MAB, Davison and Sachdev -1705.07896
Diffusion and Chaos

• Important open problem is to understand **hydrodynamic transport** (conductivity etc) in strongly coupled matter.

• In metals hydrodynamic charge/energy transport is characterized by diffusion

\[ \partial_t n = D \nabla^2 n \]

• Conductivities proportional to diffusion constants through Einstein relations e.g. for thermal diffusion \( D_T = \kappa / c_\rho \)
• In a weakly interacting system

\[ \bar{v} \quad \bar{u} \quad \bar{v} \quad \bar{u} \]

\[ D \approx \bar{v}^2 \tau \]

• Transport at strong coupling governed by

\[ \tau \approx \frac{\hbar}{k_B T} \]

Sachdev, Hartnoll

• How can we identify characteristic velocity of diffusion in strongly coupled matter?
Chaos and Scrambling

• Scrambling/chaos describes growth of operators with time

\[ C(t) = \langle [V(t), W(0)]^2 \rangle_{\beta_0} \]
• In semiclassics this grows exponentially in a chaotic system

\[ \{x(t), p(0)\} = \frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t} \implies C(t) \sim \hbar^2 e^{2\lambda_L t} \]

• Many-body quantum chaos: For interacting quantum systems with many degrees of freedom (large \( N \)) one finds

\[ C(t) \sim \frac{1}{N} e^{\lambda t} \quad t_r \ll t \ll t_s \]

\[ \lambda \leq \frac{2\pi k_B}{\hbar \beta_0} \]

Maldacena, Shenker & Stanford ('Chaos Bound')
• For operators separated in space typically get ballistic spreading

$$\langle [V(t, x), W(0, 0)]^2 \rangle_{\beta_0} \sim \frac{1}{N} e^{\lambda(t-x/v_B)}$$

• Motivation for studying these correlation functions originally came from holography/black holes and SYK models.

• However now a vast literature studying scrambling in spin chains, random circuits etc *

(In these systems the precise form of OTOC differs from the large-N results above)
Proposal

• Propose $v_B$ as characteristic velocity of diffusion

\[ D \approx v_B^2 \tau \]

• In many cases also expect Lyapunov exponent to provide measure of mean free time \( \tau \approx \tau_L = 1/\lambda \)

• Original evidence for this proposal came from studying holographic theories.
Holography

Classical gravity in asymp-AdS spacetime  →  Strongly coupled large N gauge theory

- Reissner-Nordstrom black hole
- Hawking radiation = finite temperature, $T$
- Electric field = chemical potential, $\mu$
• DC thermoelectric conductivities \( \sigma \propto \bar{K} \) can be related to geometry and fields at black hole horizon.

• Likewise chaos exponents can be calculated from gravitational shock-wave on horizon.

• Diffusion constants proportional to conductivities through Einstein relations.

MAB & Tong; Donos & Gauntlett

Shenker & Stanford; Roberts, Stanford & Susskind
• Most general result is found in relationship between energy/thermal diffusion and chaos.

• For holographic theories with dynamical exponent $z$ chaos parameters depend on temperature $T$ and microscopic variables $L$

\[ \tau_L^{-1} = 2\pi T \quad v_B^2 \sim L^2 T^{2-2/z} \]

• Nevertheless computing $D_T$ one always finds

\[ D_T = \frac{z}{2z - 2} v_B^2 \tau_L \]
Other Examples

**SYK chains**

\[
\tau_L^{-1} = 2\pi T \\
v_B^2 \sim \frac{J'^2 T}{J} \\
D_T = v_B^2 \tau_L
\]

**Critical Fermi surfaces**

Quantum chaos on a critical Fermi surface

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**Abstract**

We compute parameters characterizing many-body quantum chaos for a critical Fermi surface without quasiparticle excitations. We examine a theory of \(N\) species of fermions at non-zero density coupled to a U(1) gauge field in two spatial dimensions, and determine the Lyapunov rate and the butterfly velocity in an extended random phase approximation. The thermal diffusivity is found to be universally related to these chaos parameters i.e., the relationship is independent of \(N\), the gauge coupling constant, the Fermi velocity, the Fermi surface curvature, and high energy details.

\[
\tau_L = \frac{\hbar}{2.48 k_B T} \\
v_B \sim \frac{N v_F^{5/3}}{e^{1/3} \gamma^{1/3}} T^{1/3} \\
D_E = 0.42 v_B^2 \tau_L
\]
Part II

Hydrodynamic modes and chaos
Hydrodynamic modes and chaos

- These results constitute part of growing evidence for a more fundamental connection between chaos and energy dynamics, e.g. holography (gravity), SYK (Schwarzian)…

- In 2018 we proposed an effective field theory description of chaos where energy dynamics and exponential growth of OTOCs are governed by the same hydrodynamic d.o.f. $\sigma(t)$
• This theory predicts a precise signature of chaos in the retarded energy density two point function $G^{R}_{T00T00}(\omega, k)$.

• At low frequencies the retarded energy density two point function typically has a hydrodynamic diffusion pole

$$\omega(k) = -iD_{T}k^2 + \ldots$$

• Chaos provides a constraint on the dispersion relation for this pole at energy scales $\omega \approx i\lambda$. 
• Prediction 1: This line of poles must pass through point

\[ \omega_0 = i\lambda \quad k_0 = \frac{i\lambda}{v_B} \]

• Prediction 2: At this point the residue should vanish (`Pole-skipping`)
• These features had previously been observed numerically for the hydrodynamic mode of a holographic theory dual to an AdS5-Schwarzchild black hole.

Grozdanov, Schalm & Scopelliti (PRL)

• They can also be seen to be true for SYK chains.

Gu, Qi & Stanford

• We have now proved that pole-skipping occurs for general holographic theories.

MAB, Davison, Grozdanov & Liu `2018
• In holography $G^R_{T00T00}(\omega, k)$ is computed from studying the linearized Einstein equations about a black hole background.

• Proof relies on a surprising new feature of these equations at $(\omega_0, k_0)$.

• At this location in Fourier space the ingoing boundary condition becomes ill-posed, and there are infinitely many allowed solutions to the Einstein equations.

• From this it follows that both a line of poles and a line of zeroes pass through $(\omega_0, k_0)$. 
• Numerically one can confirm that the pole passing through this location is the hydrodynamic one

![Graphs showing Im(ω/T) vs. Im(k/T) for m/T=1 and m/T=100](image)

• We can also numerically confirm the residue of the pole vanishes as it passes through through \((ω_0, k_0)\)
Comments

• Generality of pole-skipping in holography supports our proposal that such theories are described by a type of hydro EFT.

• Key open question is to determine if such an EFT description of chaos and pole-skipping is special to maximally chaotic systems.
Thank you!