Introduction to hydrodynamics and electronic fluids

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The goal of this lecture is to give a short introduction to non-quasiparticle approaches to transport at strong coupling, e.g. hydrodynamics, memory matrices and AdS/CFT.

After setting up the stage, I will mostly focus on momentum relaxation in metallic phases.

I will also mention the possibility of fundamental bounds on transport coefficients.
Main references


Transport with long-lived quasiparticles

- Transport in a weakly-coupled metallic phase is accounted for by tracking the dynamics of the weakly-interacting quasiparticles.

- Infinite number of quasi-conserved quantities $\tau_{qp} \gg \hbar/(k_B T)$ (or $\tau_{el} \ll \tau_{inel}$).

- Kinetic Boltzmann equation: captures the dynamics of $n_{\delta k}$, the qp density at wavector $\delta k = k - k_F$. Difficulty: solving the collision integral but this is a technical obstacle, not a conceptual one.

- From the point of view of transport, this typically means that the ac conductivity

$$\sigma(\omega, k = 0) \sim \frac{\omega_p^2}{\Gamma - i\omega}, \quad \Gamma = \frac{1}{\tau_{qp}}$$

There is a sharp Drude-like peak at $\omega = 0$ and

$$\sigma_{dc} = \lim_{\omega \to 0} \sigma(\omega) = \frac{ne^2}{m} \tau_{qp} \gg \frac{1}{T}$$

This can be taken as an operational definition of a good metal.
The MIR bound

- The qp mean free path is bounded from below by Quantum Mechanics:
  \[ k_F \ell \gtrsim \hbar \]

- This implies a lower bound on the conductivity of a good metal
  \[ \sigma_{dc} = \frac{ne^2\tau_{qp}}{m} \gtrsim \frac{e^2}{\hbar} \]

- This can also be reformulated using the uncertainty principle on energy
  \[ E_F k_B T \gtrsim \hbar \quad \Rightarrow \quad \sigma_{dc} \gtrsim \frac{E_F e^2}{k_B T \hbar} \]
What about cases without long-lived quasiparticles $\tau_{qp} \sim 1/T$?

Specifically, I will focus here on cases with an emerging long lived collective mode: momentum (tomorrow, Goldstone boson as well).

Hydrodynamics: relaxation towards equilibrium $\tau \gg \tau_{th} \sim 1/T$. Expansion in small gradients which encapsulates the assumption that $\tau / \tau_{th} \gg 1$ or equivalently $\xi / \ell_{mfp} \gg 1$.

The memory matrix formalism does not assume small gradients: ‘disorder’ can vary importantly on microscopic scales. However it is only practically useful if there is only a small number of long-lived operators.

AdS/CFT gives results consistent with both previous approaches, and allows to describe the crossover from weak to strong breaking.
The starting point is conservation equations for energy (entropy), momentum and charge densities (symmetries).

\[ \partial_t s + \partial_i \left( \frac{j^i_q}{T} \right) = 0, \quad \partial_t \pi^i + \partial_j \tau^{ij} = 0, \quad \partial_t \rho + \partial_i j^i = 0 \]

Next, we add an applied electric field \( E_i \sim O(\partial) \):

\[ \partial_t \delta s + \partial_i \left( \frac{j^i_q}{T} \right) = \frac{E_i j^i}{T}, \quad \partial_t \pi^i + \partial_j \tau^{ij} = \rho \left( E^i + \nu_k F^k_i \right) \]

We give a constitutive relation to currents order by order in gradients

\[ j^i = \rho v^i - \sigma_o \left( \partial^i \mu - E^i \right) - \alpha_o \partial_i T + O(\partial^2), \]

\[ j^i_q = s T v^i - T \alpha_o \left( \partial^i \mu - E^i \right) - T \kappa_o \partial_i T + O(\partial^2), \]

\[ \tau^{ij} = \rho \delta^{ij} - \eta \left( \partial^i v^j + \partial^j v^i \right) + (\zeta - \eta) \partial_k v^k \delta^{ij} + O(\partial^2) \]
There are three longitudinal modes: two acoustic and a diffusive mode

\[ \omega_\pm = \pm c_s k - i \gamma_s k^2, \quad \omega_{\text{inc}} = -iD_{\text{inc}} k^2 \]

The sound modes are carried by momentum and pressure fluctuations

\[ G^R_{\pi \pi}, G^R_{\delta \rho \delta \rho} \sim \frac{1}{\omega^2 - c_s^2 k^2 - 2i\gamma_s k^2} \]

The diffusive mode is carried by a combination of charge and entropy

\[ G^R_{\delta \rho_{\text{inc}} \delta \rho_{\text{inc}}} \sim \frac{1}{\omega + iD_{\text{inc}} k^2}, \quad \delta \rho_{\text{inc}} = s \delta \rho - \rho \delta s \]
Finally, solve for linearized fluctuations in terms of $E_i$, using the relations between vevs and sources

$$\pi^i = \chi_{PP} v^i,$$

$$\begin{pmatrix} \delta \rho \\ \delta s \end{pmatrix} = \begin{pmatrix} \chi_{\rho \rho} & \chi_{\rho s} \\ \chi_{s \rho} & \chi_{ss} \end{pmatrix} \begin{pmatrix} \delta \mu \\ \delta T \end{pmatrix}$$

Generalized Ohm’s law

$$\begin{pmatrix} j \\ j_q \end{pmatrix} = \begin{pmatrix} \sigma & T\alpha \\ T\alpha & T\kappa \end{pmatrix} \begin{pmatrix} E \\ -\partial \delta T/T \end{pmatrix}$$
Thermoelectric conductivities

\[ \sigma(\omega) = \sigma_o + \frac{\rho^2}{\chi_{PP}} \left( \frac{i}{\omega} + \pi \delta(\omega) \right) \]

\[ \alpha(\omega) = \alpha_o + \frac{\rho s}{\chi_{PP}} \left( \frac{i}{\omega} + \pi \delta(\omega) \right) \]

\[ \tilde{\kappa}(\omega) = \tilde{\kappa}_o + \frac{s^2 T}{\chi_{PP}} \left( \frac{i}{\omega} + \pi \delta(\omega) \right) \]

- Their dc limit \( \omega \to 0 \) is formally infinite. This is due to momentum conservation and the non-zero overlap between the electric and heat currents with momentum:

\[ \chi_{JP} = \frac{\delta j^i}{\delta v^i} = \rho \]

\[ \chi_{JQP} = \frac{\delta j^i}{\delta v^i} = sT \]
Finite dc thermal conductivity

Consider the heat conductivity with open circuit boundary conditions

\[ \kappa \equiv T \frac{\delta j_q}{\delta \bar{T}} \bigg|_{j=0} = \bar{\kappa} - \frac{\alpha^2}{T \sigma} \]

Finite as \( \omega \to 0 \)

\[ \kappa = \bar{\kappa}_o - \frac{2sT}{\rho} \alpha_o + \frac{s^2 T}{\rho^2} \sigma_o \]

The open circuit boundary conditions remove the contribution of the sound modes from the thermal conductivity.

This is the thermal conductivity measured in experiments.
When we have (Galilean or Lorentz) boosts, we can fix some of the hydro coefficients.

**Galilean boosts**

\[ \chi_{PP} = mn, \quad \rho = ne, \quad \sigma_o = \alpha_o = 0 \]

\[ \pi = mj = mnev \]

**Lorentz boosts**

\[ \chi_{PP} = \epsilon + p = \mu \rho + Ts, \quad \alpha_o = -\frac{\mu}{T} \sigma_o, \quad \bar{\kappa}_o = \frac{\mu^2}{T} \sigma_o \]
Introducing weak, long wavelength disorder

Finite dc conductivities? Relax momentum, ie break translations explicitly. The simplest way to treat disorder perturbatively in a ‘mean field’ way.

\[ \partial_t \pi^i + \partial_j \tau_{ij} = -\Gamma \pi^i + \rho \left( E^i + \nu_k F^{ki} \right), \quad \Gamma \ll \Lambda \sim 1/\tau_{th} \]

The conductivity and associated resistivity become

\[ \sigma(\omega) = \sigma_o + \frac{\rho^2}{\chi_{PP}} \frac{1}{\Gamma - i\omega}, \quad \rho_{dc} = \frac{1}{\sigma_{dc}} \sim O(\Gamma) \neq 0 \]

Disorder is a (dangerously) irrelevant deformation for the resistivity.
Hydrodynamic signatures in electronic flows

- Wiedemann-Franz law for conventional metals
  \[ \mathcal{L} = \frac{\kappa_e}{\sigma T} = \frac{\pi^2}{3} \left( \frac{k_B^2}{e} \right)^2 \equiv \mathcal{L}_0 \]

  The WF law holds because both \( \kappa_e, \sigma \sim \tau_{qp} \).

- In very clean Graphene near the charge neutrality point
  \[ \kappa_e = \frac{\chi_{PP}}{T \Gamma}, \quad \sigma \sim \sigma_o \quad \Rightarrow \quad \mathcal{L} \sim O \left( \frac{1}{\Gamma} \right) \gg \mathcal{L}_0 \]

[Crossno et al, Science 351 6277 (2016)]
Other recent experiments

- Backflows and negative resistance in Graphene due to viscous effects

- Viscous contributions to the resistance in restricted channels in PdCoO$_2$
  [Moll et al, Science 351 2016].

- Viscous contributions to the resistance, violations of WF law and Hall measurements in WP$_2$

- Dirk van der Marel’s talk on Thursday.
Beyond simple hydrodynamics

- How do we compute $\sigma_o, \Gamma$?
- Short-scale disorder?
- Strong disorder?

Need a more microscopic approach!